Gov 50: 23. Inference with Mathematical Models

Matthew Blackwell

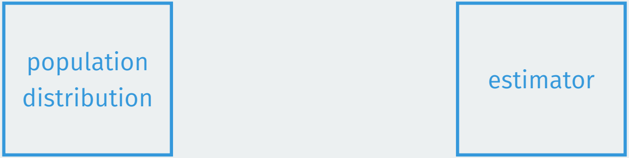
Harvard University

Roadmap

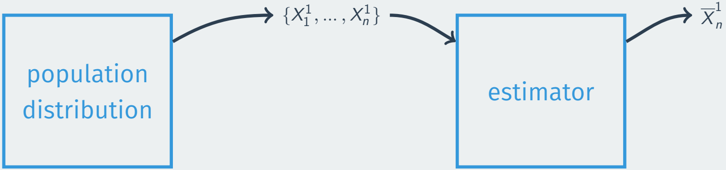
1. Central limit theorem
2. Normal distribution
3. Using the Normal for inference

1/ Central limit theorem

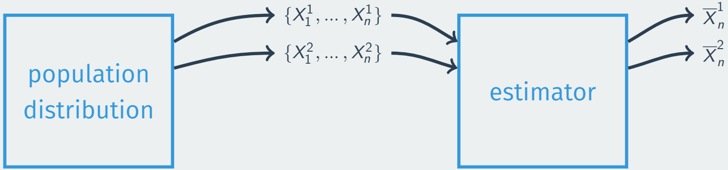
Sampling distribution, in pictures



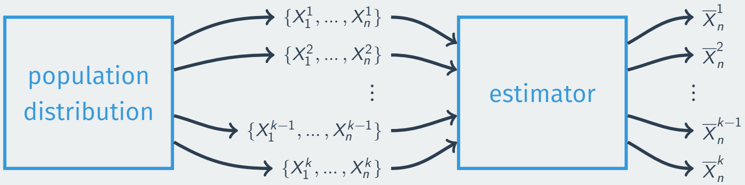
Sampling distribution, in pictures



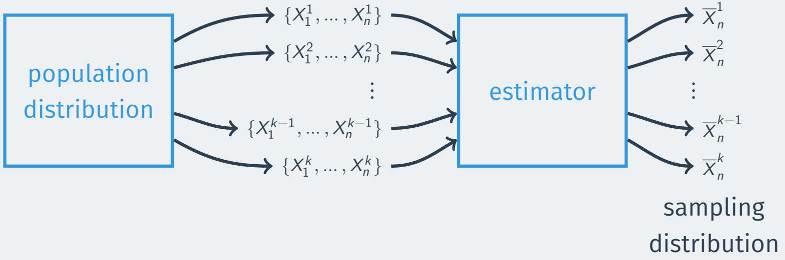
Sampling distribution, in pictures



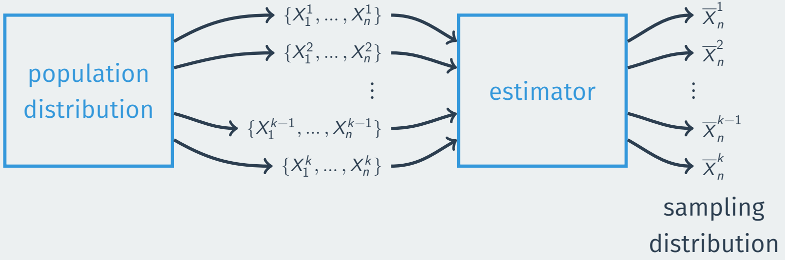
Sampling distribution, in pictures



Sampling distribution, in pictures



Sampling distribution, in pictures



Sampling distribution of the sample proportion

$$\text{sample mean} = \text{population mean} + \text{chance error}$$
$$\overline{X} = \mu + \text{chance error}$$

Sampling distribution of the sample proportion

$$\text{sample mean} = \text{population mean} + \text{chance error}$$
$$\overline{X} = \mu + \text{chance error}$$

Then $\overline{X}$ centered at μ.

Sampling distribution of the sample proportion

$$\text{sample mean} = \text{population mean} + \text{chance error}$$
$$\overline{X} = \mu + \text{chance error}$$

Then $\overline{X}$ centered at μ.

Spread: standard deviation of the sampling distribution is the **standard error**

Sampling distribution of the sample proportion

$$\text{sample mean} = \text{population mean} + \text{chance error}$$
$$\overline{X} = \mu + \text{chance error}$$

Then $\overline{X}$ centered at μ.

Spread: standard deviation of the sampling distribution is the **standard error**

Spread of the sample mean

- **Standard error**: how big is the chance error on average?

Spread of the sample mean

- **Standard error**: how big is the chance error on average?
    - This is the standard deviation of the estimator **across repeated samples**.

Spread of the sample mean

- **Standard error**: how big is the chance error on average?
    - This is the standard deviation of the estimator **across repeated samples**.
    - With random samples, we can get a formula for the SE for many estimators.

Spread of the sample mean

- **Standard error**: how big is the chance error on average?
  - This is the standard deviation of the estimator **across repeated samples**.
  - With random samples, we can get a formula for the SE for many estimators.
- Standard error for the sample mean:

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{\text{population standard deviation}}{\sqrt{\text{sample size}}}$$

Spread of the sample mean

- **Standard error**: how big is the chance error on average?
  - This is the standard deviation of the estimator **across repeated samples**.
  - With random samples, we can get a formula for the SE for many estimators.
- Standard error for the sample mean:

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{\text{population standard deviation}}{\sqrt{\text{sample size}}}$$

- Two components:

Spread of the sample mean

- **Standard error**: how big is the chance error on average?
    - This is the standard deviation of the estimator **across repeated samples**.
    - With random samples, we can get a formula for the SE for many estimators.
- Standard error for the sample mean:

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{\text{population standard deviation}}{\sqrt{\text{sample size}}}$$

- Two components:
    - Population SD: more spread of the variable in the population → more spread of sample means

Spread of the sample mean

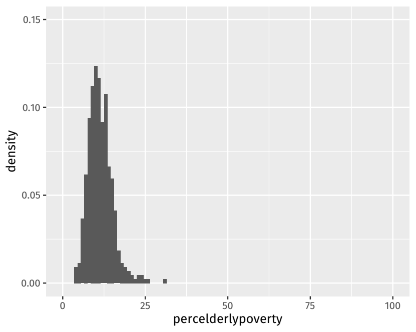
- **Standard error**: how big is the chance error on average?
    - This is the standard deviation of the estimator **across repeated samples**.
    - With random samples, we can get a formula for the SE for many estimators.
- Standard error for the sample mean:

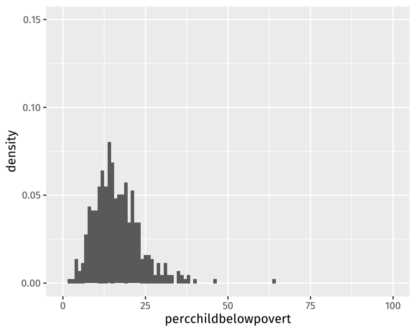
$$SE = \frac{\sigma}{\sqrt{n}} = \frac{\text{population standard deviation}}{\sqrt{\text{sample size}}}$$

- Two components:
    - Population SD: more spread of the variable in the population → more spread of sample means
    - Size of the sample: larger sample → smaller spread of the sample means

Midwest counties

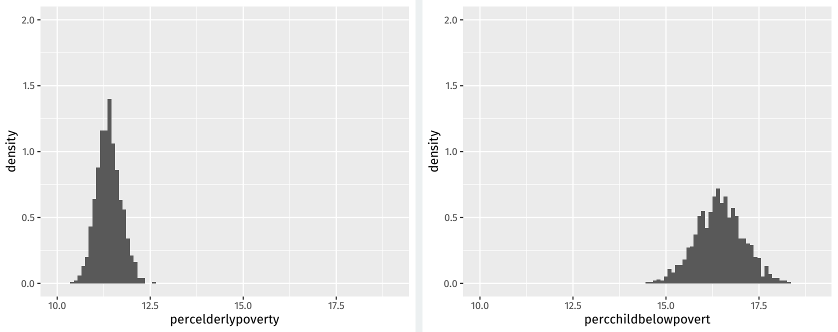
Population distributions:





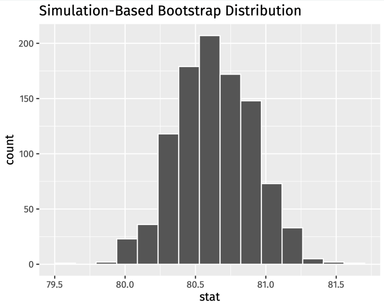
Midwest counties

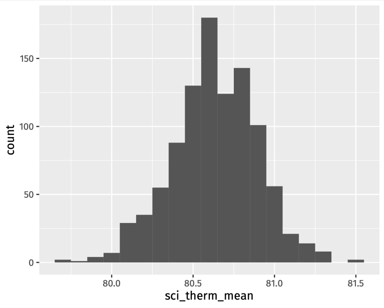
Sampling distributions with $n = 100$

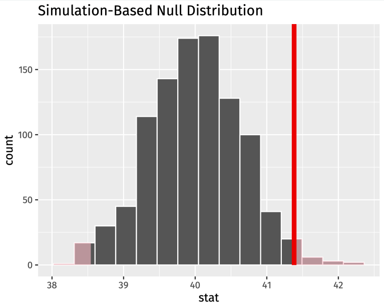


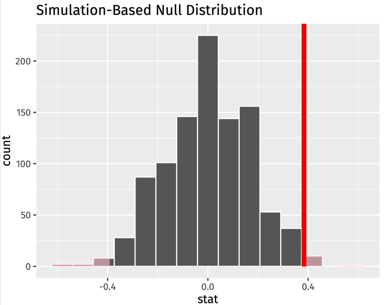
More population spread → higher SE

Similarity in the bootstrap/null distributions

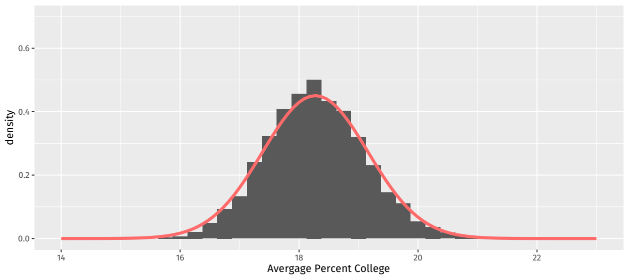






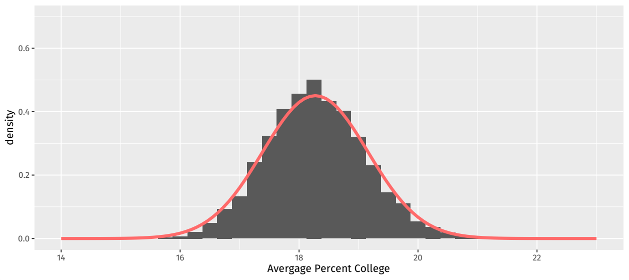


Conditions for the CLT



Central limit theorem: sums and means of **random samples** tend to be normally distributed as the **sample size grows**.

Conditions for the CLT

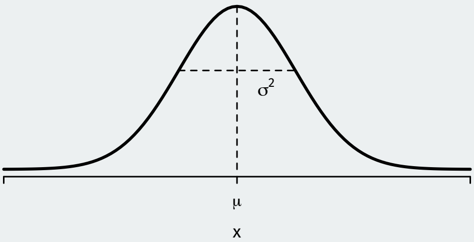


Central limit theorem: sums and means of **random samples** tend to be normally distributed as the **sample size grows**.

Many, many estimators will follow the CLT and have a normal distribution and will be easier to use this to do inference rather than doing increasingly complicated simulations.

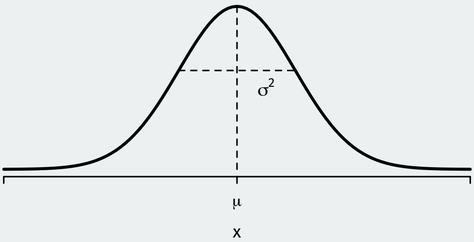
2/ Normal distribution

Normal distribution



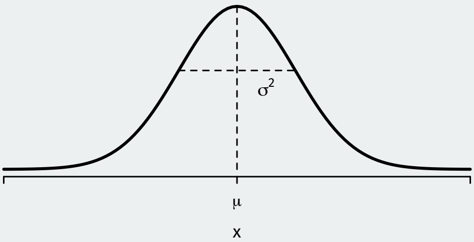
- A normal distribution can be affect by two values:

Normal distribution



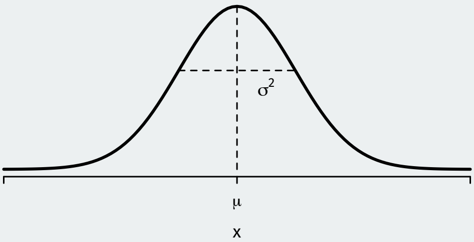
- A normal distribution can be affect by two values:
    - **mean/expected value** usually written as μ

Normal distribution



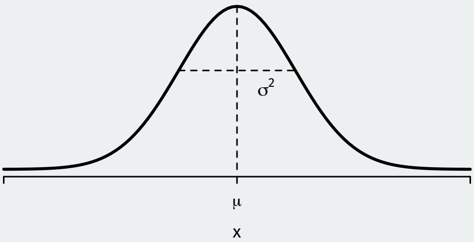
- A normal distribution can be affect by two values:
  - **mean/expected value** usually written as μ
  - **variance** written as σ^2 (standard deviation is σ)

Normal distribution



- A normal distribution can be affect by two values:
    - **mean/expected value** usually written as μ
    - **variance** written as σ^2 (standard deviation is σ)
    - Written $X \sim N(\mu, \sigma^2)$.

Normal distribution



- A normal distribution can be affect by two values:
  - **mean/expected value** usually written as μ
  - **variance** written as σ^2 (standard deviation is σ)
  - Written $X \sim N(\mu, \sigma^2)$.
- **Standard normal distribution**: mean 0 and standard deviation 1.

Reentering and scaling the normal

- How do transformations of a normal work?

Reentering and scaling the normal

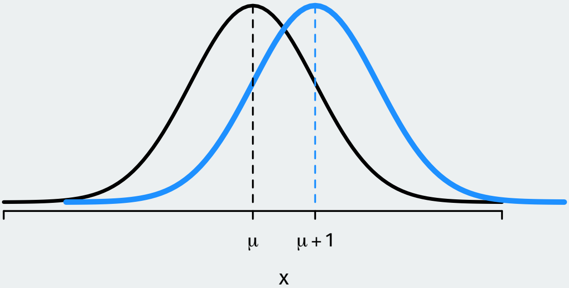
- How do transformations of a normal work?
- Let $X \sim N(\mu, \sigma^2)$ and c be a constant.

Reentering and scaling the normal

- How do transformations of a normal work?
- Let $X \sim N(\mu, \sigma^2)$ and c be a constant.
- If $Z = X + c$, then $Z \sim N(\mu + c, \sigma^2)$.

Reentering and scaling the normal

- How do transformations of a normal work?
- Let $X \sim N(\mu, \sigma^2)$ and c be a constant.
- If $Z = X + c$, then $Z \sim N(\mu + c, \sigma^2)$.
- Intuition: adding a constant to a normal shifts the distribution by that constant.



Recentering and scaling the normal

- Let $X \sim N(\mu, \sigma^2)$ and c be a constant.

Recentering and scaling the normal

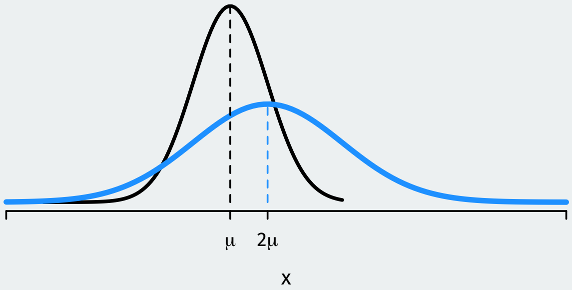
- Let $X \sim N(\mu, \sigma^2)$ and c be a constant.
- If $Z = cX$, then $Z \sim N(c\mu, (c\sigma)^2)$.

Recentering and scaling the normal

- Let $X \sim N(\mu, \sigma^2)$ and c be a constant.
- If $Z = cX$, then $Z \sim N(c\mu, (c\sigma)^2)$.
- Intuition: multiplying a normal by a constant scales the mean and the variance.

Recentering and scaling the normal

- Let $X \sim N(\mu, \sigma^2)$ and c be a constant.
- If $Z = cX$, then $Z \sim N(c\mu, (c\sigma)^2)$.
- Intuition: multiplying a normal by a constant scales the mean and the variance.



Z-scores of normals

- These facts imply the **z-score** of a normal variable is a standard normal:

$$z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

Z-scores of normals

- These facts imply the **z-score** of a normal variable is a standard normal:

$$z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

- Subtract the mean and divide by the SD $\rightsquigarrow$ standard normal.

Z-scores of normals

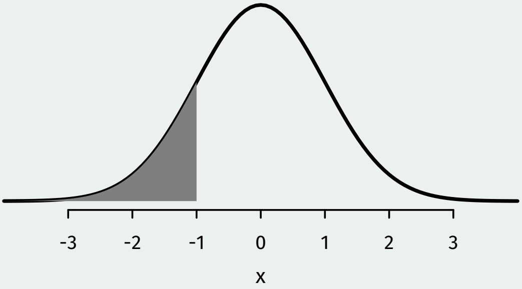
- These facts imply the **z-score** of a normal variable is a standard normal:

$$z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

- Subtract the mean and divide by the SD $\rightsquigarrow$ standard normal.
- z-score measures how many SDs away from the mean a value of X is.

Normal probability calculations

What's the probability of being below -1 for a standard normal?



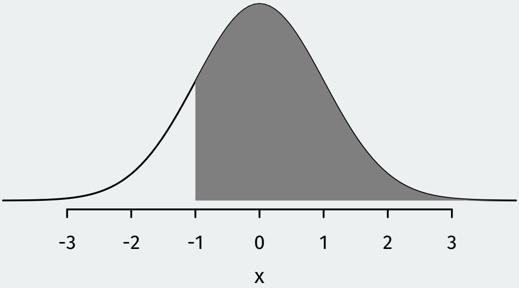
This is the area under the normal curve, which `pnorm( )` function gives us this:

```
pnorm(-1, mean = 0, sd = 1)
```

```
## [1] 0.159
```

Normal probability calculations

What's the probability of being **above** -1 for a standard normal?



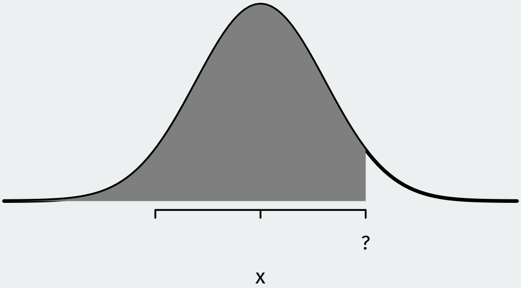
Total area under the curve (1) minus the area below -1:

```
1 - pnorm(-1, mean = 0, sd = 1)
```

```
## [1] 0.841
```

Normal quantiles

What if we want to know the opposite? What value of the normal distribution puts 95% of the distribution below it?



This is a **quantile** and we can get it using qnorm( ):

```
qnorm(0.95, mean = 0, sd = 1)
```

```
## [1] 1.64
```

3/ Using the Normal for inference

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?
- Latest Gallup poll:

How popular is Joe Biden?

- What proportion of the public approves of Biden’s job as president?
- Latest Gallup poll:
  - Sept 1st-16th

How popular is Joe Biden?

- What proportion of the public approves of Biden’s job as president?
- Latest Gallup poll:
    - Sept 1st-16th
    - 812 adult Americans

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?
- Latest Gallup poll:
    - Sept 1st-16th
    - 812 adult Americans
    - Telephone interviews

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?
- Latest Gallup poll:
    - Sept 1st-16th
    - 812 adult Americans
    - Telephone interviews
    - Approve (42%), Disapprove (56%)

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?
- Latest Gallup poll:
  - Sept 1st-16th
  - 812 adult Americans
  - Telephone interviews
  - Approve (42%), Disapprove (56%)
- Define r.v. Y_i for Biden approval:

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?
- Latest Gallup poll:
  - Sept 1st-16th
  - 812 adult Americans
  - Telephone interviews
  - Approve (42%), Disapprove (56%)
- Define r.v. Y_i for Biden approval:
  - $Y_i = 1 \rightsquigarrow$ respondent i approves of Biden, 0 otherwise.

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?
- Latest Gallup poll:
    - Sept 1st-16th
    - 812 adult Americans
    - Telephone interviews
    - Approve (42%), Disapprove (56%)
- Define r.v. Y_i for Biden approval:
    - $Y_i = 1 \rightsquigarrow$ respondent i approves of Biden, 0 otherwise.
    - $p = \mathbb{P}(Y_i = 1)$ the population proportion of Biden approvers.

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?
- Latest Gallup poll:
    - Sept 1st-16th
    - 812 adult Americans
    - Telephone interviews
    - Approve (42%), Disapprove (56%)
- Define r.v. Y_i for Biden approval:
    - $Y_i = 1 \rightsquigarrow$ respondent i approves of Biden, 0 otherwise.
    - $p = \mathbb{P}(Y_i = 1)$ the population proportion of Biden approvers.
    - $\overline{Y} = 0.42$ is the sample proportion.

Standard errors for sample proportions

How variable will our sample proportion be? Depends on the **standard error**.

Standard errors for sample proportions

How variable will our sample proportion be? Depends on the **standard error**.

Special rule for SEs of sample proportion $\overline{Y}$:

$$SE \text{ for } \overline{Y} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{(\text{pop. proportion}) \times (1 - \text{pop. proportion})}{\text{sample size}}}$$

Standard errors for sample proportions

How variable will our sample proportion be? Depends on the **standard error**.

Special rule for SEs of sample proportion $\overline{Y}$:

$$SE \text{ for } \overline{Y} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{(\text{pop. proportion}) \times (1 - \text{pop. proportion})}{\text{sample size}}}$$

Because we don't know p, we replace it with our best guess, $\overline{Y}$:

$$\widehat{SE} = \sqrt{\frac{\overline{Y}(1-\overline{Y})}{n}}$$

CLT for confidence intervals

$$\overline{Y} - p = \text{chance error}$$

- How can we figure out a range of plausible chance errors?

CLT for confidence intervals

$$\overline{Y} - p = \text{chance error}$$

- How can we figure out a range of plausible chance errors?
  - Find a range of plausible chance errors and add them to $\overline{Y}$

CLT for confidence intervals

$$\overline{Y} - p = \text{chance error}$$

- How can we figure out a range of plausible chance errors?
    - Find a range of plausible chance errors and add them to $\overline{Y}$
    - With **bootstrap**, we used resampling to simulate chance error.

CLT for confidence intervals

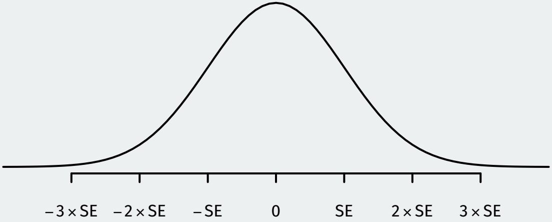
$$\overline{Y} - p = \text{chance error}$$

- How can we figure out a range of plausible chance errors?
  - Find a range of plausible chance errors and add them to $\overline{Y}$
  - With **bootstrap**, we used resampling to simulate chance error.
- Central limit theorem implies

$$\overline{Y} \approx N\left(p, \frac{p(1-p)}{n}\right)$$

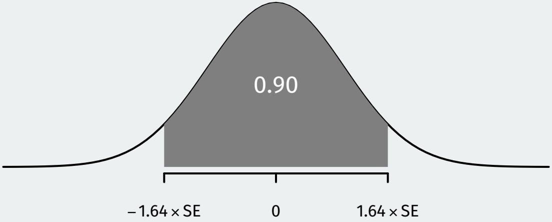
Chance error: $\overline{Y} - p$ is approximately normal with mean 0 and SE equal to $\sqrt{p(1-p)/n}$

Chance errors



If $\overline{Y} \sim N(p, SE^2)$, then chance errors are $\overline{Y} - p \sim N(0, SE^2)$ so:

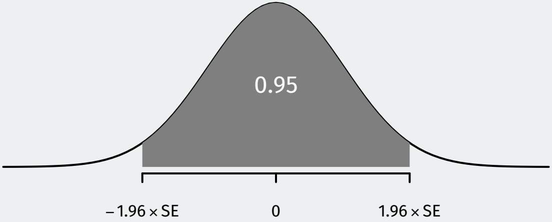
Chance errors



If $\overline{Y} \sim N(p, SE^2)$, then chance errors are $\overline{Y} - p \sim N(0, SE^2)$ so:

- ≈ 90% of chance errors $\overline{Y} - p$ are within 1.64 SEs of the mean.

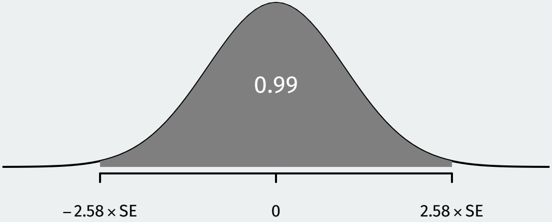
Chance errors



If $\overline{Y} \sim N(p, SE^2)$, then chance errors are $\overline{Y} - p \sim N(0, SE^2)$ so:

- ≈ 90% of chance errors $\overline{Y} - p$ are within 1.64 SEs of the mean.
- ≈ 95% of chance errors $\overline{Y} - p$ are within 1.96 SEs of the mean.

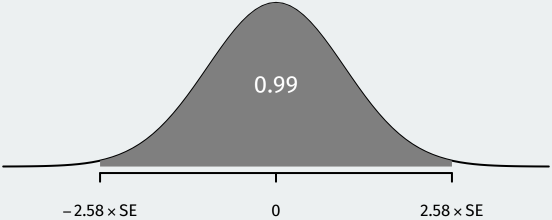
Chance errors



If $\overline{Y} \sim N(p, SE^2)$, then chance errors are $\overline{Y} - p \sim N(0, SE^2)$ so:

- ≈ 90% of chance errors $\overline{Y} - p$ are within 1.64 SEs of the mean.
- ≈ 95% of chance errors $\overline{Y} - p$ are within 1.96 SEs of the mean.
- ≈ 99% of chance errors $\overline{Y} - p$ are within 2.58 SEs of the mean.

Chance errors



If $\overline{Y} \sim N(p, SE^2)$, then chance errors are $\overline{Y} - p \sim N(0, SE^2)$ so:

- ≈ 90% of chance errors $\overline{Y} - p$ are within 1.64 SEs of the mean.
- ≈ 95% of chance errors $\overline{Y} - p$ are within 1.96 SEs of the mean.
- ≈ 99% of chance errors $\overline{Y} - p$ are within 2.58 SEs of the mean.

This implies we can build a 95% confidence interval with $\overline{Y} \pm 1.96 \times SE$

How did we get those values?

- First, choose a **confidence level**.

How did we get those values?

- First, choose a **confidence level**.
    - What percent of chance errors do you want to count as “plausible”?

How did we get those values?

- First, choose a **confidence level**.
    - What percent of chance errors do you want to count as “plausible”?
    - Convention is 95%.

How did we get those values?

- First, choose a **confidence level**.
    - What percent of chance errors do you want to count as “plausible”?
    - Convention is 95%.
- $100 \times (1 - \alpha)\%$ confidence interval:

$$CI = \overline{Y} \pm z_{\alpha/2} \times SE$$

How did we get those values?

- First, choose a **confidence level**.
  - What percent of chance errors do you want to count as "plausible"?
  - Convention is 95%.
- $100 \times (1 - \alpha)\%$ confidence interval:

$$CI = \overline{Y} \pm z_{\alpha/2} \times SE$$

  - In polling, $\pm z_{\alpha/2} \times SE$ is called the **margin of error**

How did we get those values?

- First, choose a **confidence level**.
  - What percent of chance errors do you want to count as “plausible”?
  - Convention is 95%.
- $100 \times (1 - \alpha)\%$ confidence interval:

$$CI = \overline{Y} \pm z_{\alpha/2} \times SE$$

  - In polling, $\pm z_{\alpha/2} \times SE$ is called the **margin of error**
- $z_{\alpha/2}$ is the $N(0, 1)$ z-score that would put $\alpha/2$ in the upper tail:

How did we get those values?

- First, choose a **confidence level**.
    - What percent of chance errors do you want to count as "plausible"?
    - Convention is 95%.
- $100 \times (1 - \alpha)\%$ confidence interval:

$$CI = \overline{Y} \pm z_{\alpha/2} \times SE$$

    - In polling, $\pm z_{\alpha/2} \times SE$ is called the **margin of error**
- $z_{\alpha/2}$ is the $N(0,1)$ z-score that would put $\alpha/2$ in the upper tail:
    - $\mathbb{P}(-z_{\alpha/2} < Z < z_{\alpha/2}) = \alpha$

How did we get those values?

- First, choose a **confidence level**.
  - What percent of chance errors do you want to count as “plausible”?
  - Convention is 95%.
- $100 \times (1 - \alpha)\%$ confidence interval:

$$CI = \overline{Y} \pm z_{\alpha/2} \times SE$$

  - In polling, $\pm z_{\alpha/2} \times SE$ is called the **margin of error**
- $z_{\alpha/2}$ is the $N(0, 1)$ z-score that would put $\alpha/2$ in the upper tail:
  - $\mathbb{P}(-z_{\alpha/2} < Z < z_{\alpha/2}) = \alpha$
  - 90% CI $\rightsquigarrow \alpha = 0.1 \rightsquigarrow z_{\alpha/2} = 1.64$

How did we get those values?

- First, choose a **confidence level**.
    - What percent of chance errors do you want to count as "plausible"?
    - Convention is 95%.
- $100 \times (1 - \alpha)\%$ confidence interval:

$$CI = \overline{Y} \pm z_{\alpha/2} \times SE$$

    - In polling, $\pm z_{\alpha/2} \times SE$ is called the **margin of error**
- $z_{\alpha/2}$ is the $N(0, 1)$ z-score that would put $\alpha/2$ in the upper tail:
    - $\mathbb{P}(-z_{\alpha/2} < Z < z_{\alpha/2}) = \alpha$
    - 90% CI $\rightsquigarrow \alpha = 0.1 \rightsquigarrow z_{\alpha/2} = 1.64$
    - 95% CI $\rightsquigarrow \alpha = 0.05 \rightsquigarrow z_{\alpha/2} = 1.96$

How did we get those values?

- First, choose a **confidence level**.
    - What percent of chance errors do you want to count as "plausible"?
    - Convention is 95%.
- $100 \times (1 - \alpha)\%$ confidence interval:

$$CI = \overline{Y} \pm z_{\alpha/2} \times SE$$

    - In polling, $\pm z_{\alpha/2} \times SE$ is called the **margin of error**
- $z_{\alpha/2}$ is the $N(0,1)$ z-score that would put $\alpha/2$ in the upper tail:
    - $\mathbb{P}(-z_{\alpha/2} < Z < z_{\alpha/2}) = \alpha$
    - 90% CI $\rightsquigarrow \alpha = 0.1 \rightsquigarrow z_{\alpha/2} = 1.64$
    - 95% CI $\rightsquigarrow \alpha = 0.05 \rightsquigarrow z_{\alpha/2} = 1.96$
    - 99% CI $\rightsquigarrow \alpha = 0.01 \rightsquigarrow z_{\alpha/2} = 2.58$

Standard normal z-scores in R

`qnorm(x, lower.tail = FALSE)` will find the quantile of $N(0, 1)$ that puts x in the upper tail:

```
qnorm(0.05, lower.tail = FALSE)
```

Standard normal z-scores in R

`qnorm(x, lower.tail = FALSE)` will find the quantile of $N(0, 1)$ that puts x in the upper tail:

```
qnorm(0.05, lower.tail = FALSE)
```

```
## [1] 1.64
```

Standard normal z-scores in R

`qnorm(x, lower.tail = FALSE)` will find the quantile of $N(0, 1)$ that puts x in the upper tail:

```r
qnorm(0.05, lower.tail = FALSE)
```

```
## [1] 1.64
```

```r
qnorm(0.025, lower.tail = FALSE)
```

Standard normal z-scores in R

`qnorm(x, lower.tail = FALSE)` will find the quantile of $N(0, 1)$ that puts x in the upper tail:

```
qnorm(0.05, lower.tail = FALSE)
```

```
## [1] 1.64
```

```
qnorm(0.025, lower.tail = FALSE)
```

```
## [1] 1.96
```

Standard normal z-scores in R

`qnorm(x, lower.tail = FALSE)` will find the quantile of $N(0, 1)$ that puts x in the upper tail:

```
qnorm(0.05, lower.tail = FALSE)
```

```
## [1] 1.64
```

```
qnorm(0.025, lower.tail = FALSE)
```

```
## [1] 1.96
```

```
qnorm(0.005, lower.tail = FALSE)
```

Standard normal z-scores in R

`qnorm(x, lower.tail = FALSE)` will find the quantile of $N(0,1)$ that puts x in the upper tail:

```r
qnorm(0.05, lower.tail = FALSE)
```

```
## [1] 1.64
```

```r
qnorm(0.025, lower.tail = FALSE)
```

```
## [1] 1.96
```

```r
qnorm(0.005, lower.tail = FALSE)
```

```
## [1] 2.58
```